Trigonometry:

Derivation of Formula!

Sum and Difference of Two Angles

Engr. HB

TRIGONOMETRY – FORMULA SHEET

Functions of Right Triangle

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c} = (\text{soh})$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c} = (\text{cah})$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b} = (\text{toa})$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{a} = (\text{cho})$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{b} = (\text{sha})$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{b}{a} = (\text{cao})$$

Phytagorean Theorem

$$a^2 + b^2 = c^2$$

Phytagorean Relations

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

Half Angle Formulas

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

Powers of Functions

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\tan^2 x = \frac{1-\cos\theta}{1+\cos\theta}$$

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

$$\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$$

$$\sin^4 x = \frac{1}{8}(\cos 4x - 4\cos 2x + 3)$$

$$\cos^4 x = \frac{1}{8}(\cos 4x + 4\cos 2x + 3)$$

Triple Angle Formulas

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$\cos 3x = 4\cos^3 x - 3\cos x$$

$$\tan 3x = \frac{3\tan x - 4\tan^3 x}{1 - 3\tan^2 x}$$

For Oblique Triangle

Sine Law

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine Law

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Tangent Law

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$$

$$\frac{b-c}{b+c} = \frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}$$

$$\frac{c-a}{c+a} = \frac{\tan\left(\frac{C-A}{2}\right)}{\tan\left(\frac{C+A}{2}\right)}$$

Angle Measurement

$360° = 2\pi$

$360° = 1$ revolution

$360° = 400$ grads

$360° = 6400$ mils

Area of Triangle:

Given base b and altitude h

$A = \frac{1}{2}(b)(h)$

Given two sides a&b, included angle θ

$A = \frac{1}{2}(a)(b)(\sin\theta)$

Given 3 sides a,b,c (Heron's Formula)

$$\sqrt{(s)(s-a)(s-b)(s-c)}$$

Given 3 sides A,B,C and one side a:

$$A = \frac{a^2 \sin B \sin C}{2\sin A}$$

Median of Triangle:

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

$$m_b = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2}$$

$$m_c = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$$

Altitude of Triangle:

$$a_a = \frac{2AT}{a}$$

$$a_b = \frac{2AT}{b}$$

$$a_c = \frac{2AT}{c}$$

Sum & Difference of Two Angles

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

$$\sin(x-y) = \sin x\cos y - \cos x\sin y$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$$

Sum & Difference of Functions

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\tan x + \tan y = \frac{\sin(x+y)}{\cos x\cos y}$$

$$\tan x - \tan y = \frac{\sin(x-y)}{\cos x\cos y}$$

Mollweide's Equations

$$\frac{a-b}{c} = \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{C}{2}\right)}$$

$$\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

Length of Angle Bisector

$$b_a = \frac{2}{(b+c)}\sqrt{bcs(s-a)}$$

$$b_b = \frac{2}{(a+c)}\sqrt{acs(s-b)}$$

$$b_c = \frac{2}{(a+b)}\sqrt{abs(s-c)}$$

Double Angle Formulas

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\cos 2x = 2\cos^2 x - 1$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Product of Functions

$$\sin x\cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\sin x\sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos x\cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

Angles

Acute angle: angle < 90°

Right angle: angle = 90°

Obtuse angle: angle > 90°

Straight angle: angle = 180°

Reflex angle: angle > 90°

s, (semi perimeter) $= \frac{a+b+c}{2}$

Complementary angle: are angles whose sum is 90°

Supplementary angle: are angles whose sum is 180°

Explementary angle: are angles whose sum is 360°

TRIGONOMETRY – FORMULA SHEET

intersection of medians – "centroid"

intersection of altitudes – "orthocenter"

intersection of angle bisector – "incenter"

Phytagorean Triple Generator!

$$a = m^2 - n^2$$

$$b = 2mn$$

$$c = m^2 + n^2$$

Such that a,b,c is a positive integer

$$m > n > 0$$

Spherical Triangle

$$180° < (A + B + C) < 540°$$

Area of radius R

$$A = \frac{\pi R^2 E}{180°}$$

Spherical Excess

$$E = A + B + C - 180°$$

$$\tan\frac{E}{4} = \sqrt{\tan\frac{s}{2}\tan\frac{s-a}{2}\tan\frac{s-b}{2}\tan\frac{s-c}{2}}$$

where $s = \dfrac{(a + b + c)}{2}$

Spherical Defect = $360° - (a + b + c)$

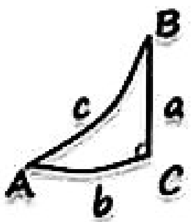
1 minute of arc = 1 nautical mile

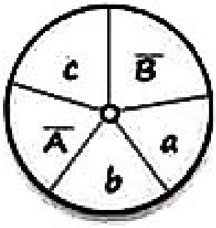
1 nautical mile = 6080 feet

1 statute mile = 5280 feet

1 knot = 1 nautical mile per hour

Right Spherical Triangle





$$\bar{A} = 90° - A$$

$$\bar{B} = 90° - B$$

$$\bar{c} = 90° - c$$

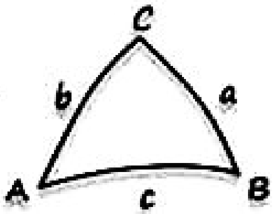
SIN-TAAD rule

The sine of any middle part is equal to the product of the tangents of its adjacent parts.

SIN-COOP rule

The sine of any middle part is equal to the product of the cosines of its opposite parts.

Oblique Spherical Triangle



Law of Sines

$$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C}$$

Law of Cosines for Side

$$\cos a = (\cos b)(\cos c) + (\sin b)(\sin c)(\cos A)$$

$$\cos b = (\cos a)(\cos c) + (\sin a)(\sin c)(\cos B)$$

$$\cos c = (\cos a)(\cos b) + (\sin a)(\sin b)(\cos C)$$

Law of Cosines for Angles

$$\cos A = -(\cos B)(\cos C) + (\sin B)(\sin C)(\cos a)$$

$$\cos B = -(\cos A)(\cos C) + (\sin A)(\sin C)(\cos b)$$

$$\cos C = -(\cos A)(\cos B) + (\sin A)(\sin B)(\cos c)$$

Napier's Analogies

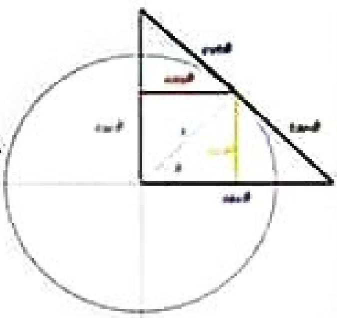
$$\frac{\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)} = \frac{\tan\left(\frac{a-b}{2}\right)}{\tan\left(\frac{c}{2}\right)}$$

$$\frac{\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)} = \frac{\tan\left(\frac{a+b}{2}\right)}{\tan\left(\frac{c}{2}\right)}$$

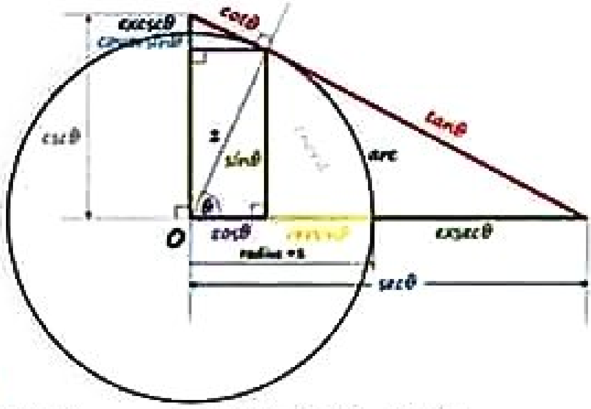
$$\frac{\sin\left(\frac{a-b}{2}\right)}{\sin\left(\frac{a+b}{2}\right)} = \frac{\tan\left(\frac{A-B}{2}\right)}{\cot\left(\frac{C}{2}\right)}$$

$$\frac{\cos\left(\frac{a-b}{2}\right)}{\cos\left(\frac{a+b}{2}\right)} = \frac{\tan\left(\frac{A+B}{2}\right)}{\cot\left(\frac{C}{2}\right)}$$

The six trigonometric function in a unit circle!



The secret trigonometric functions!!



$$\text{versin}\,\theta = 1 - \cos\theta$$

$$\text{vercos}\,\theta = 1 + \cos\theta$$

$$\text{coversin}\,\theta = 1 - \sin\theta$$

$$\text{covercos}\,\theta = 1 + \sin\theta$$

$$\text{exsec}\,\theta = \sec\theta - 1$$

$$\text{excsc}\,\theta = \csc\theta - 1$$

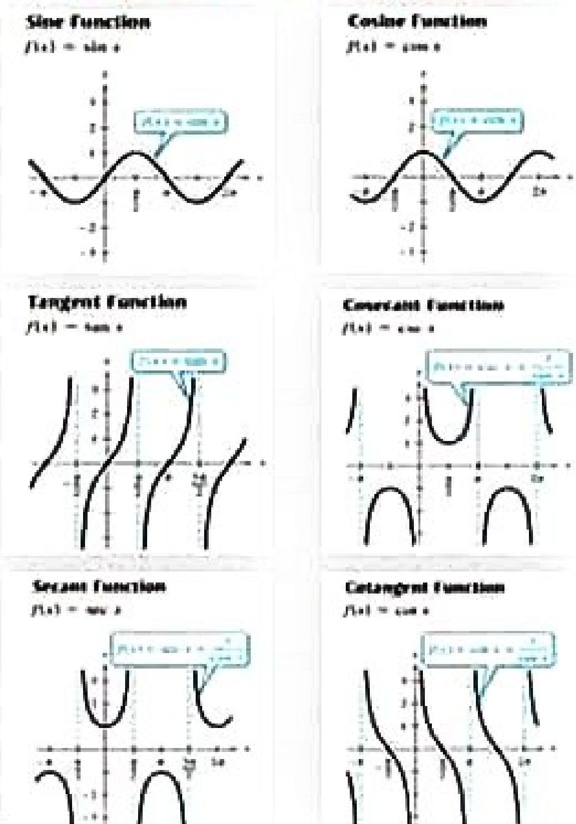
$$\text{haversin}\,\theta = \frac{1}{2}(1 - \cos\theta)$$

$$\text{havercos}\,\theta = \frac{1}{2}(1 + \cos\theta)$$

$$\text{hacoversin}\,\theta = \frac{1}{2}(1 - \sin\theta)$$

$$\text{hacovercos}\,\theta = \frac{1}{2}(1 + \sin\theta)$$

Graph of Trigonometric Functions



TRIGONOMETRY: DERIVATION OF FORMULAS:

DERIVATION OF SINE LAW

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

DERIVATION OF COSINE LAW

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

DERIVATION OF TANGENT LAW

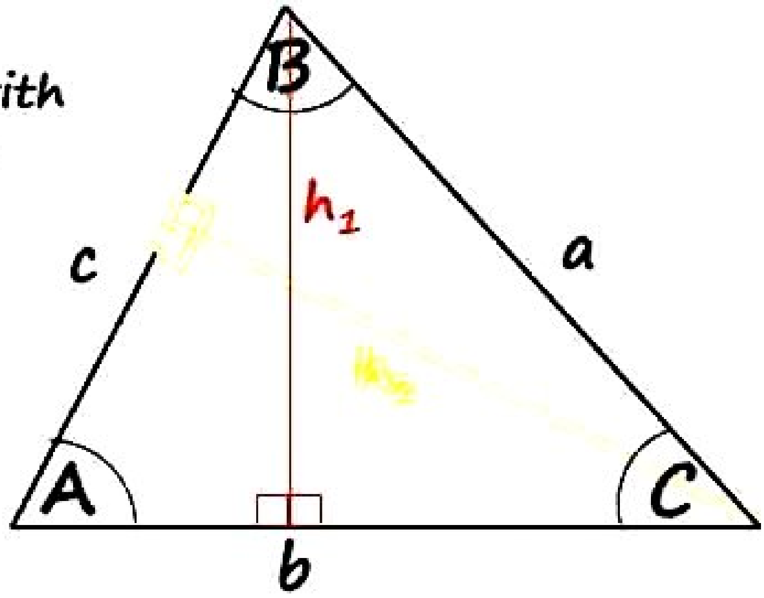
$$\frac{a-b}{a+b} = \left(\frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}\right)$$

$$\frac{b-c}{b+c} = \left(\frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}\right)$$

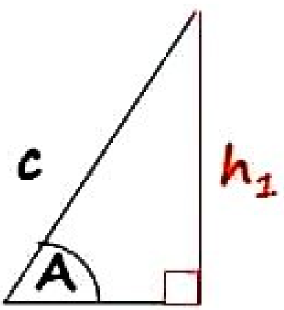
$$\frac{c-a}{c+a} = \left(\frac{\tan\left(\frac{C-A}{2}\right)}{\tan\left(\frac{C+A}{2}\right)}\right)$$

DERIVATION OF SINE LAW

Consider this oblique triangle with angle A,B,C and side a,b and c



For this triangle:

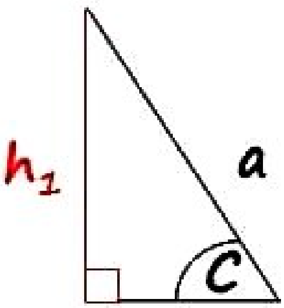


$$\sin\theta = \frac{opp}{hyp}$$

$$\sin A = \frac{h_1}{c}$$

$$h_1 = (\sin A)(c) \longrightarrow \text{Eq.1}$$

For this triangle:



$$\sin\theta = \frac{opp}{hyp}$$

$$\sin C = \frac{h_1}{a}$$

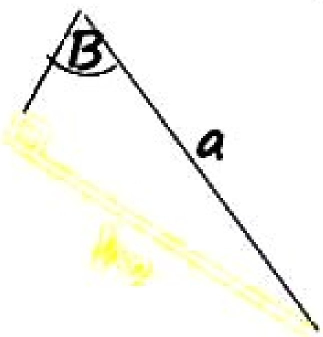
$$h_1 = (\sin C)(a) \longrightarrow \text{Eq.2}$$

Solving Eq 1 and 2 =

$$(\sin A)(c) = (\sin C)(a)$$

$$\boxed{\frac{\sin A}{a} = \frac{\sin C}{c}}$$

For this triangle:



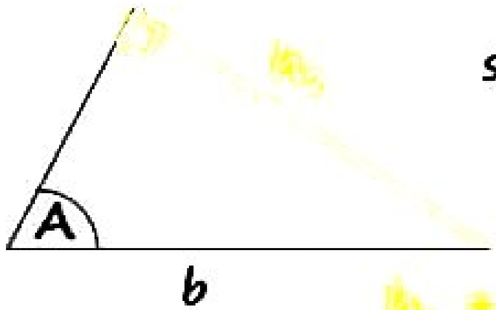
$$\sin\theta = \frac{opp}{hyp}$$

$$\sin B = \frac{h_2}{a}$$

$$h_2 = (\sin B)(a)$$

Eq.3

For this triangle:



$$\sin\theta = \frac{opp}{hyp}$$

$$\sin A = \frac{h_2}{b}$$

$$h_2 = (\sin A)(b)$$

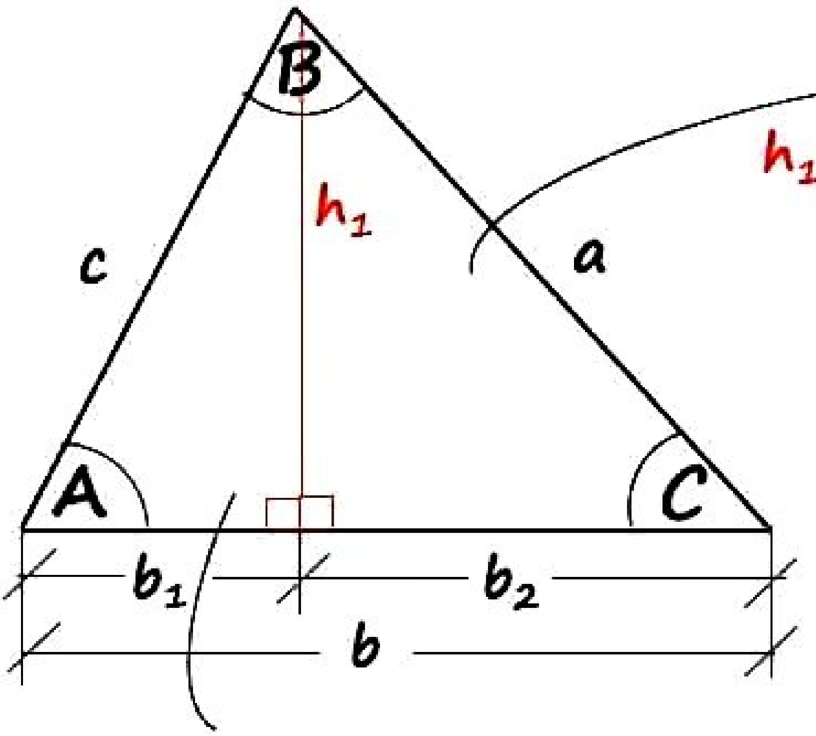
Eq.4

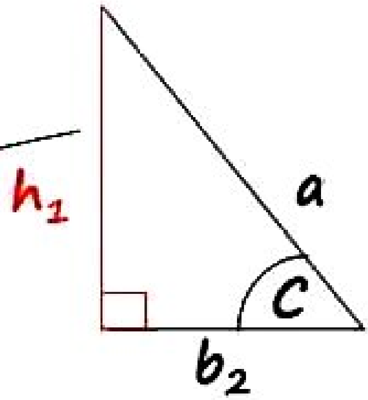
Solving Eq 3 and 4 $(\sin B)(a) = (\sin A)(b)$

$$\boxed{\frac{\sin B}{b} = \frac{\sin A}{a}} \longrightarrow \boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$

DERIVATION OF COSINE LAW

Consider this oblique triangle with angle A,B,C and side a,b and c





$$\sin\theta = \frac{opp}{hyp}$$

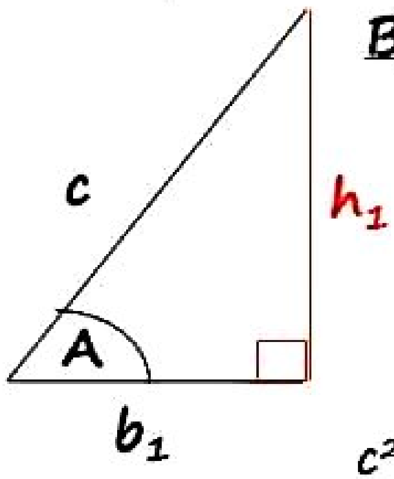
$$\sin C = \frac{h_1}{a}$$

$$h_1 = (\sin C)(a)$$

$$\cos\theta = \frac{adj}{hyp}$$

$$\cos C = \frac{b_2}{a}$$

$$b_2 = (\cos C)(a)$$



By Phytagorean Theorem:

$$c^2 = (b_1)^2 + (h_1)^2 \qquad \text{but } b = b_1 + b_2$$

$$c^2 = (b - b_2)^2 + (h_1)^2$$

$$h_1 = (\sin C)(a) \qquad b_2 = (\cos C)(a)$$

$$c^2 = (b - a\cos C)^2 + (a\sin C)^2$$

$$c^2 = (b^2 - 2ab\cos C + a^2\cos^2 C) + a^2\sin^2 C$$

$$c^2 = a^2\cos^2 C + a^2\sin^2 C + b^2 - 2ab\cos C$$

By identities: $\sin^2 + \cos^2 = 1$

$$c^2 = a^2(\cos^2 C + \sin^2 C) + b^2 - 2ab\cos C$$

$$\boxed{\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ a^2 &= b^2 + c^2 - 2bc\cos A \end{aligned}}$$

DERIVATION OF TANGENT LAW

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$$

By Sine Law: $\longrightarrow$ $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$$\frac{\sin A}{a} = \frac{\sin B}{b} \longrightarrow a = \frac{(b)\sin A}{\sin B}$$

$$a - b = \frac{(b)\sin A}{\sin B} - b \longrightarrow a - b = b\left(\frac{\sin A}{\sin B} - 1\right)$$

$$a + b = \frac{(b)\sin A}{\sin B} + b \longrightarrow a + b = b\left(\frac{\sin A}{\sin B} + 1\right)$$

$$\frac{a-b}{a+b} = \frac{\cancel{b}\left(\frac{\sin A}{\sin B} - 1\right)}{\cancel{b}\left(\frac{\sin A}{\sin B} + 1\right)} = \frac{\left(\frac{\sin A}{\sin B} - 1\right)}{\left(\frac{\sin A}{\sin B} + 1\right)} = \frac{\left(\frac{\sin A - \sin B}{\cancel{\sin B}}\right)}{\left(\frac{\sin A + \sin B}{\cancel{\sin B}}\right)}$$

$$\frac{a-b}{a+b} = \left(\frac{\sin A - \sin B}{\sin A + \sin B}\right)$$

By Sum & Difference of Functions:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\frac{a-b}{a+b} = \left(\frac{\sin A - \sin B}{\sin A + \sin B}\right) = \frac{\cancel{2}\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{\cancel{2}\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$$

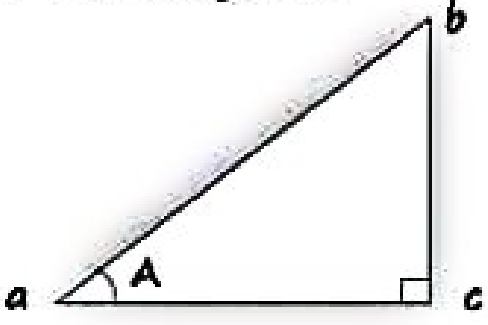
$$\frac{a-b}{a+b} = \cot\left(\frac{A+B}{2}\right)\tan\left(\frac{A-B}{2}\right)$$

$$\frac{a-b}{a+b} = \left(\frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}\right) \qquad \frac{b-c}{b+c} = \left(\frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}\right) \qquad \frac{c-a}{c+a} = \left(\frac{\tan\left(\frac{C-A}{2}\right)}{\tan\left(\frac{C+A}{2}\right)}\right)$$

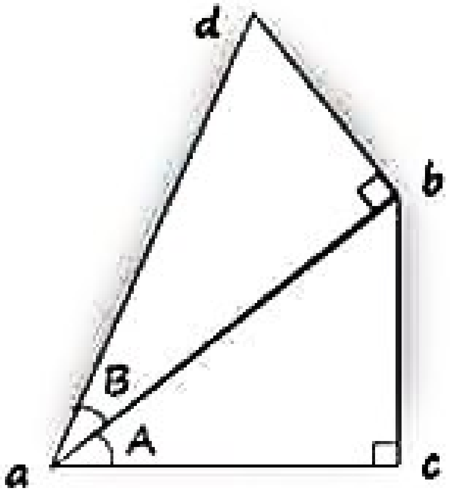
Derivation of Formula

Sum and Difference of Two Angles

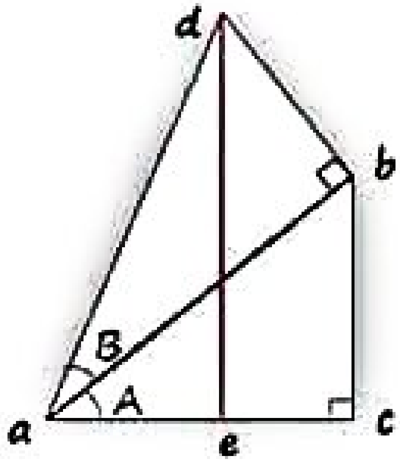
1. Draw a right triangle with sides a, b, c and angle A



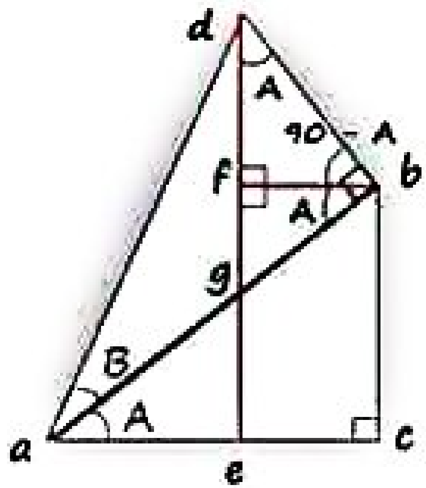
2. Draw another right triangle with sides a, d, b and angle B, starting at point b making its adjacent the hypotenuse of triangle abc



3. Make a straight red line from point d perpendicular to the side ac



4. Project a red line from point b perpendicular to the straight line d-e



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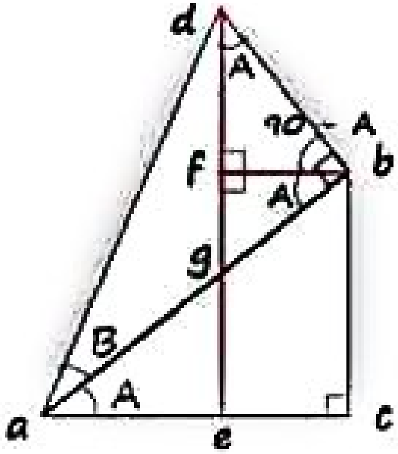
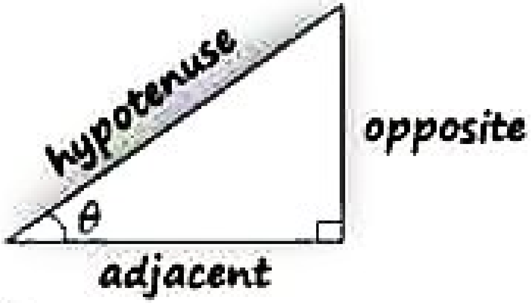
Derivation of Formula

Sum and Difference of Two Angles

Let this figure be our basis on formula derivation:



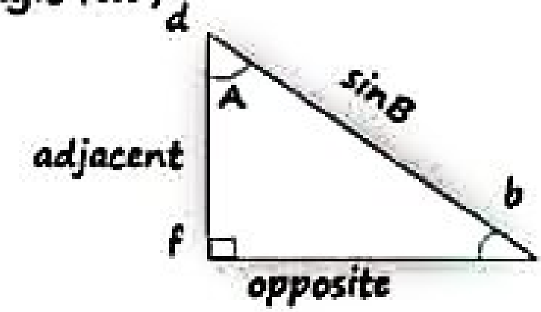
For a regular right triangle; we have:



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}};\ \text{opposite} = (\text{hypotenuse})(\sin\theta)$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}};\ \text{adjacent} = (\text{hypotenuse})(\cos\theta)$$

For triangle fdb,



$$\theta = A$$

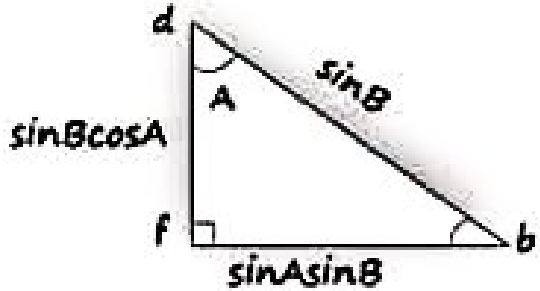
$$\text{opposite} = (\text{hypotenuse})(\sin\theta)$$

$$\text{opposite} = (\sin B)(\sin A) = \sin A \sin B$$

$$\text{adjacent} = (\text{hypotenuse})(\cos\theta)$$

$$\text{adjacent} = (\sin B)(\cos A) = \sin B \cos A$$

Thus, triangle adb will become:



DERIVE

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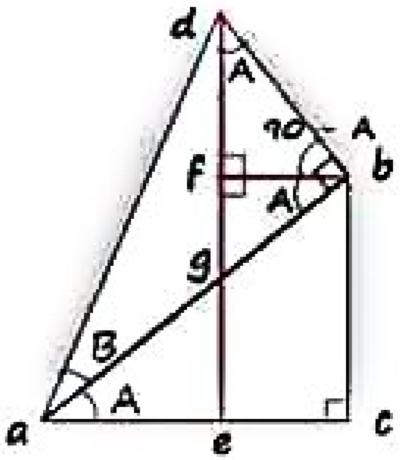
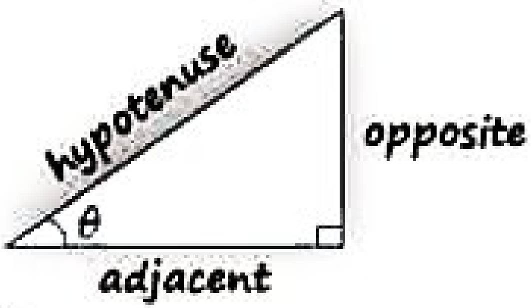
Derivation of Formula

Sum and Difference of Two Angles

Let this figure be our basis on formula derivation:



For a regular right triangle; we have:

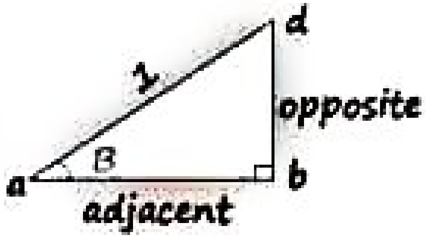


$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$; opposite = (hypotenuse)($\sin\theta$)

$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$; adjacent = (hypotenuse)($\cos\theta$)

DERIVE

For triangle adb, let ad = 1 (hypotenuse)



$\theta = B$

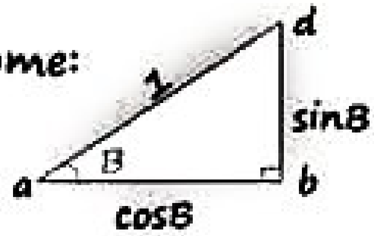
opposite = (hypotenuse)($\sin\theta$)

opposite = (1)($\sin B$) = $\sin B$

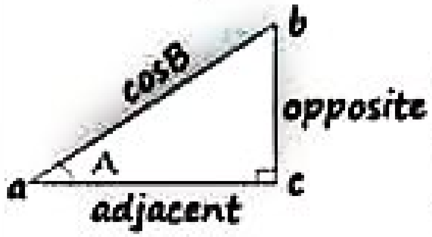
adjacent = (hypotenuse)($\cos\theta$)

adjacent = (1)($\cos B$) = $\cos B$

Thus, triangle adb will become:



For triangle abc,



$\theta = A$

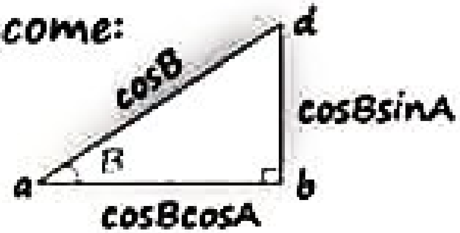
opposite = (hypotenuse)($\sin\theta$)

opposite = ($\cos B$)($\sin A$)

adjacent = (hypotenuse)($\cos\theta$)

adjacent = ($\cos B$)($\cos A$)

Thus, triangle abc will become:



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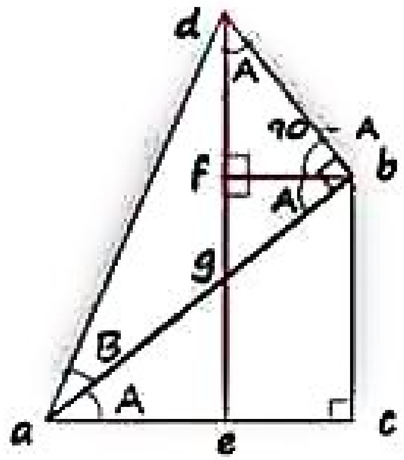
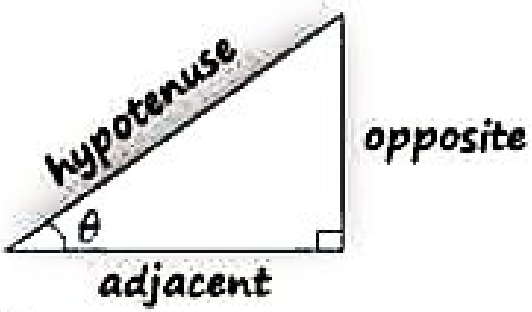
Derivation of Formula

Sum and Difference of Two Angles

Let this figure be our basis on formula derivation:



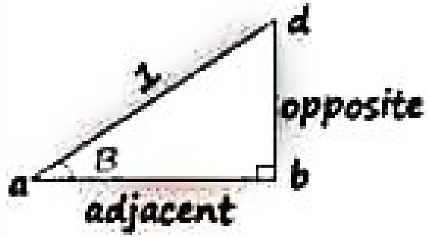
For a regular right triangle; we have:



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}};\ \text{opposite} = (\text{hypotenuse})(\sin\theta)$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}};\ \text{adjacent} = (\text{hypotenuse})(\cos\theta)$$

For triangle adb, let ad = 1 (hypotenuse)



$\theta = B$

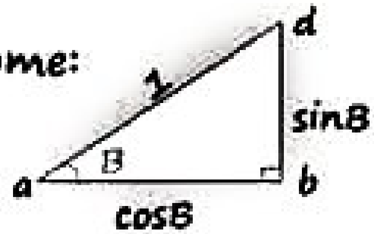
opposite = (hypotenuse)($\sin\theta$)

opposite = (1)($\sin B$) = $\sin B$

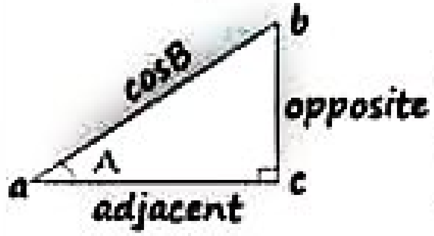
adjacent = (hypotenuse)($\cos\theta$)

adjacent = (1)($\cos B$) = $\cos B$

Thus, triangle adb will become:



For triangle abc,



$\theta = A$

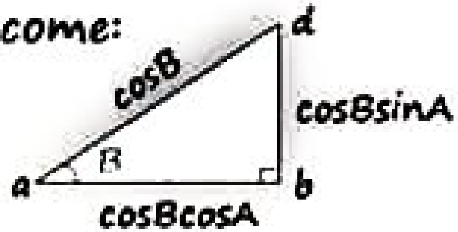
opposite = (hypotenuse)($\sin\theta$)

opposite = ($\cos B$)($\sin A$)

adjacent = (hypotenuse)($\cos\theta$)

adjacent = ($\cos B$)($\cos A$)

Thus, triangle abc will become:



DERIVE

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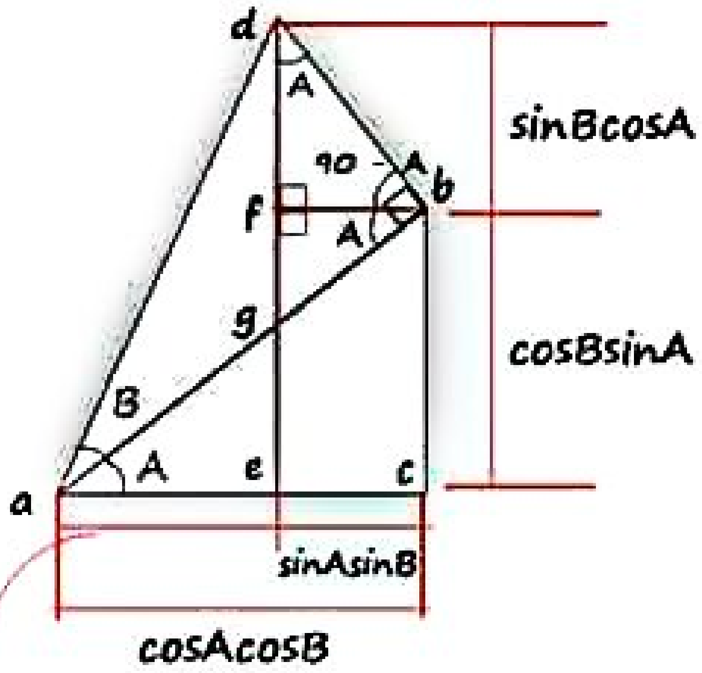
Derivation of Formula

Sum and Difference of Two Angles

Thus, it will become:



cosAcosB - sinAsinB

For a regular right triangle; we have:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}};$$

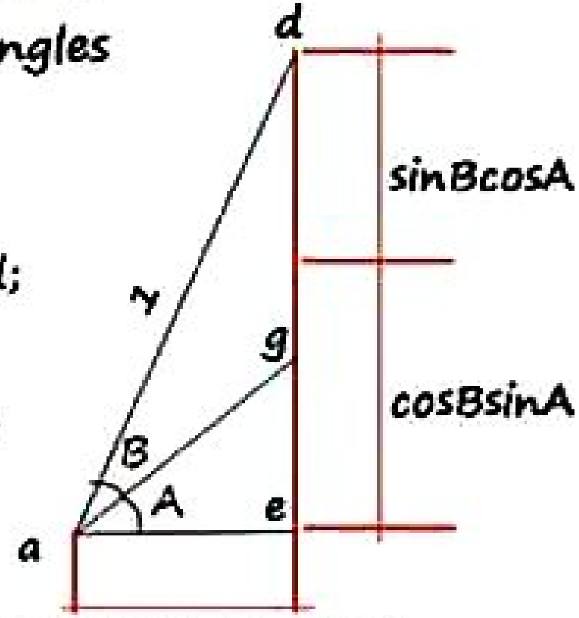
opposite opposite = (hypotenuse)(sinθ)

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}};$$

adjacent = (hypotenuse)(cosθ)

For triangle aed;

$\theta = A + B$



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}};$$

$$\sin(A + B) = \frac{\sin A\cos B + \sin B\cos A}{1};$$

$$\boxed{\sin(A + B) = \sin A\cos B + \sin B\cos A}$$ ✓

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}};$$

$$\cos(A + B) = \frac{\cos A\cos B - \sin A\sin B}{1};$$

$$\boxed{\cos(A + B) = \cos A\cos B - \sin A\sin B}$$ ✓

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Derivation of Formula

Sum and Difference of Two Angles

For tangent function;

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)};$$

$$\tan(A+B) = \frac{\sin A\cos B + \sin B\cos A}{\cos A\cos B - \sin A\sin B};$$

Divide all by "cosAcosB";

$$\tan(A+B) = \frac{\dfrac{\sin A\cancel{\cos B}}{\cos A\cancel{\cos B}} + \dfrac{\sin B\cancel{\cos A}}{\cancel{\cos A}\cos B}}{\dfrac{\cancel{\cos A\cos B}}{\cancel{\cos A\cos B}} - \dfrac{\sin A\sin B}{\cos A\cos B}};$$

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}}$$

For difference of 2 angles,

Let $B = -B$ and note that $\sin(-B) = -\sin B$

$\cos(-B) = \cos B$ and $\tan(-B) = -\tan B$

Thus

$$\sin(A-B) = \sin A\cos B - \sin B\cos A$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Summary!

$$\sin(A+B) = \sin A\cos B + \sin B\cos A$$

$$\sin(A-B) = \sin A\cos B - \sin B\cos A$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

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TRIGONOMETRY

TRIPLE ANGLE IDENTITIES

DERIVATION OF FORMULA

- $\boxed{\sin(3\theta) = 3\sin\theta - 4\sin^3\theta}$ ✓
- $\boxed{\cos(3\theta) = 4\cos^3\theta - 3\cos\theta}$ ✓
- $\boxed{\tan(3\theta) = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}}$ ✓

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Prove that $\tan(3\theta) = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

Let $\tan(3\theta) = \tan(\theta + 2\theta)$

From double angle identities ; $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Let $A = \theta$; $B = 2\theta$

$$\therefore \tan(\theta + 2\theta) = \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta \tan 2\theta}$$

From D.A.I ; $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

$$\therefore \tan(\theta + 2\theta) = \frac{\tan\theta + \dfrac{2\tan\theta}{1-\tan^2\theta}}{1 - \tan\theta\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)}$$

$$= \frac{\dfrac{\tan\theta(1-\tan^2\theta) + 2\tan\theta}{1-\tan^2\theta}}{\dfrac{(1-\tan^2\theta) - \tan\theta(2\tan\theta)}{1-\tan^2\theta}}$$

$$= \frac{\tan\theta - \tan^3\theta + 2\tan\theta}{1 - \tan^2\theta - 2\tan^2\theta}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$\boxed{\tan(3\theta) = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}}$$ //

Prove that $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ //

Let $\cos(3\theta) = \cos(\theta + 2\theta)$

From double angle identities; $\underline{\cos(A+B) = \cos A\cos B - \sin A\sin B}$

Let $A = \theta$; $B = 2\theta$

$\cos(\theta + 2\theta) = \cos\theta\cos 2\theta - \sin\theta\sin 2\theta$

But ; $\underline{\cos 2\theta = \cos^2\theta - \sin^2\theta}$ $\underline{\sin 2\theta = 2\sin\theta\cos\theta}$

Then ; $\cos(\theta + 2\theta) = \cos\theta(\cos^2\theta - \sin^2\theta) - \sin\theta(2\sin\theta\cos\theta)$

$= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta$

; $\underline{\sin^2\theta + \cos^2\theta = 1}$; $\underline{\sin^2\theta = 1 - \cos^2\theta}$

$= \cos^3\theta - (1-\cos^2\theta)\cos\theta - 2(1-\cos^2\theta)\cos\theta$

$= \cos^3\theta - \cos\theta + \cos^3\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

$$\boxed{\cos(3\theta) = 4\cos^3\theta - 3\cos\theta}$$ // ✓

Prove that $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$ //

Let $\sin(3\theta) = \sin(\theta + 2\theta)$

From double angle identities; $\underline{\sin(A+B) = \sin A\cos B + \cos A\sin B}$

Let $A = \theta$; $B = 2\theta$

$\sin(3\theta) = \sin(\theta + 2\theta) = \sin\theta\cos 2\theta + \cos\theta\sin 2\theta$

But $\underline{\cos 2\theta = 1 - 2\sin^2\theta}$ and $\underline{\sin 2\theta = 2\sin\theta\cos\theta}$

Then; $\sin(\theta + 2\theta) = \sin\theta(1 - 2\sin^2\theta) + \cos\theta(2\sin\theta\cos\theta)$

$= \sin\theta - 2\sin^3\theta + 2\sin\theta\cos^2\theta$

But $\underline{\sin^2\theta + \cos^2\theta = 1}$; $\underline{\cos^2\theta = 1 - \sin^2\theta}$

Then; $= \sin\theta - 2\sin^3\theta + 2\sin\theta(1 - \sin^2\theta)$

$= \sin\theta - 2\sin^3\theta + 2\sin\theta - 2\sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

$$\boxed{\sin(3\theta) = 3\sin\theta - 4\sin^3\theta}$$ // ✓

Engr. HB

DERIVATION – LENGTH OF ANGLE BISECTOR OF TRIANGLE

Thus, equation 3 will become:

$$b^2\left(\frac{ac}{b+c}\right) + c^2\left(\frac{ab}{b+c}\right) = (a)\left(b_a^{\,2} + \left(\frac{ac}{b+c}\right)\left(\frac{ab}{b+c}\right)\right)$$

$$\frac{ab^2c}{b+c} + \frac{abc^2}{b+c} = (a)\left(b_a^{\,2} + \frac{a^2bc}{(b+c)2}\right)$$

$$\frac{(abc)(b+c)}{b+c} = (a)\left(b_a^{\,2} + \frac{a^2bc}{(b+c)2}\right)$$

$$bc = \left(b_a^{\,2} + \frac{a^2bc}{(b+c)2}\right)$$

$$b_a^{\,2} = bc - \frac{a^2bc}{(b+c)2}$$

$$b_a^{\,2} = bc\left(1 - \frac{a^2}{(b+c)2}\right)$$

$$b_a^{\,2} = bc\left(\frac{(b+c)2 - a^2}{(b+c)2}\right)$$

By difference of 2 squares:

$$(b+c)^2 - a^2 = ((b+c)-a)((b+c)+a)$$

$$b_a^{\,2} = bc\left(\frac{((b+c)-a)((b+c)+a)}{(b+c)2}\right)$$

$$b_a^{\,2} = bc\left[\frac{(a+b+c)(a+b+c-2a)}{(b+c)2}\right]$$

But $a + b + c = 2s$, Thus:

$$b_a^{\,2} = bc\left[\frac{(a+b+c)(a+b+c-2a)}{(b+c)2}\right]$$

$$b_a^{\,2} = bc\left[\frac{(2s)(2s-2a)}{(b+c)2}\right]$$

$$b_a^{\,2} = \frac{(bc)(2s)(2)(s-a)}{(b+c)2}$$

$$b_a^{\,2} = \frac{4bcs(s-a)}{(b+c)2}$$

$$b_a = \sqrt{\frac{4bcs(s-a)}{(b+c)2}}$$

$$b_a = \frac{2}{(b+c)}\sqrt{bcs(s-a)}$$

$$\boxed{b_a = \frac{2}{(b+c)}\sqrt{bcs(s-a)}}$$

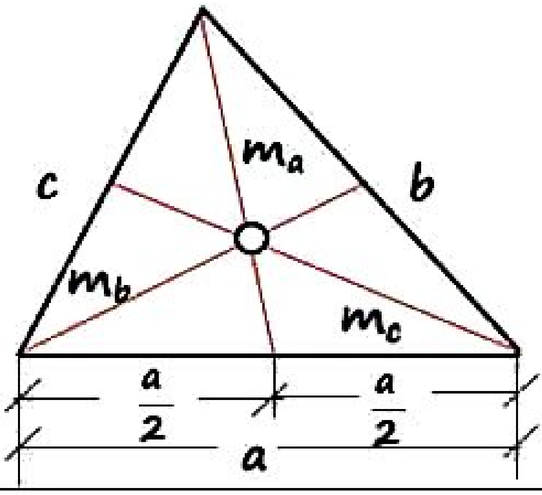
$$\boxed{b_b = \frac{2}{(a+c)}\sqrt{acs(s-b)}}$$

$$\boxed{b_c = \frac{2}{(a+b)}\sqrt{abs(s-c)}}$$

where s, (semi perimeter) $= \dfrac{a+b+c}{2}$

TRIGONOMETRY: DERIVATION OF FORMULAS:

DERIVATION OF MEDIAN OF TRIANGLE

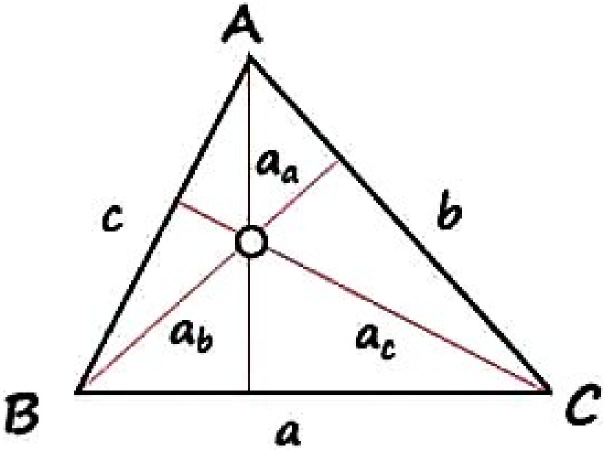


$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

$$m_b = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2}$$

$$m_c = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$$

DERIVATION OF ALTITUDE OF TRIANGLE

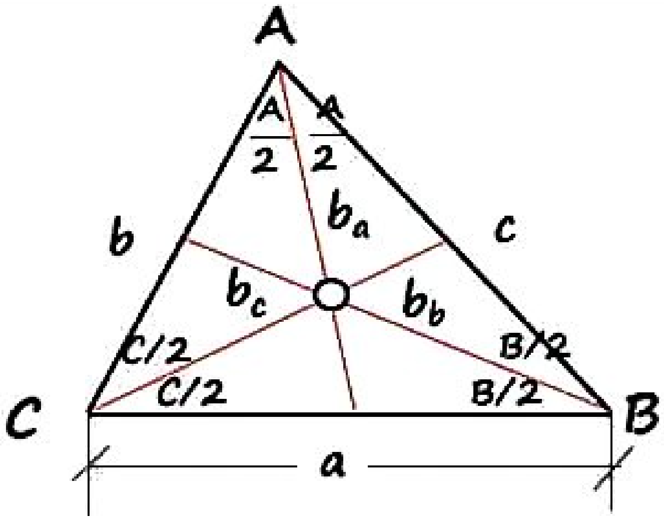


$$a_a = \frac{2A_T}{a}$$

$$a_b = \frac{2A_T}{b}$$

$$a_c = \frac{2A_T}{c}$$

DERIVATION - LENGTH OF ANGLE BISECTOR OF TRIANGLE



$$b_a = \frac{2}{(b+c)}\sqrt{bcs(s-a)}$$

$$b_b = \frac{2}{(a+c)}\sqrt{acs(s-b)}$$

$$b_c = \frac{2}{(a+b)}\sqrt{abs(s-c)}$$

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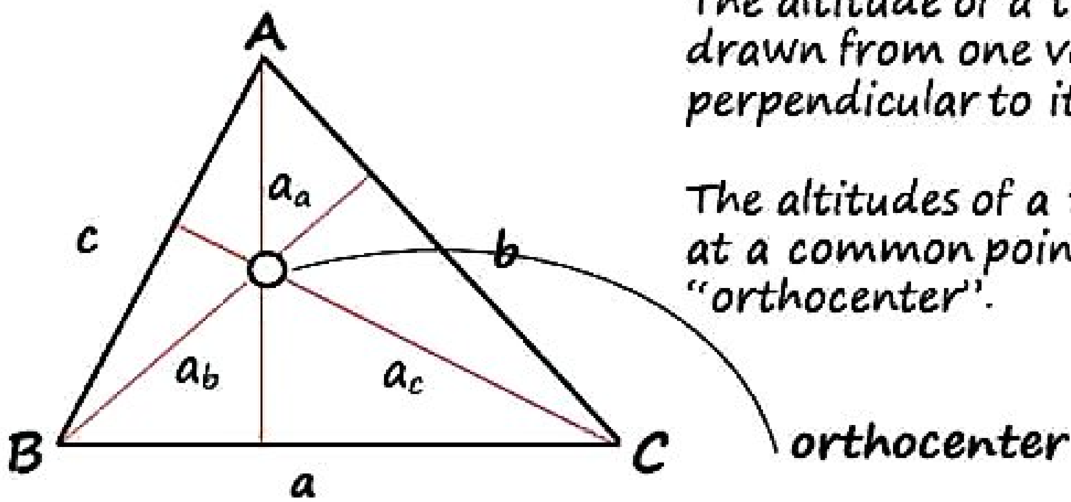
DERIVATION OF ALTITUDE OF TRIANGLE

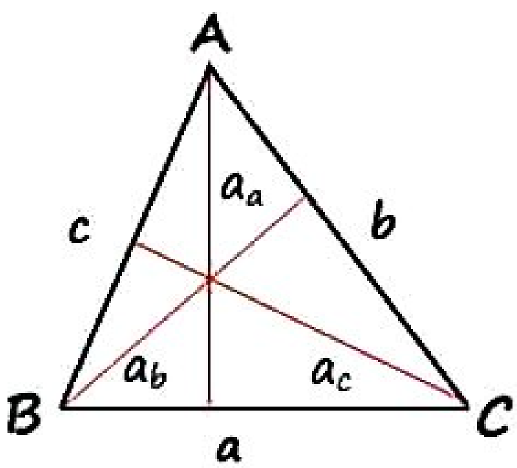


The altitude of a triangle is the line drawn from one vertex perpendicular to its opposite side.

The altitudes of a triangle intersect at a common point called the "orthocenter".

Let A_T = area of the triangle

Given base and height, area of triangle = A_T = ½(base)(height)



$$A_T = \tfrac{1}{2}(b)(h)$$

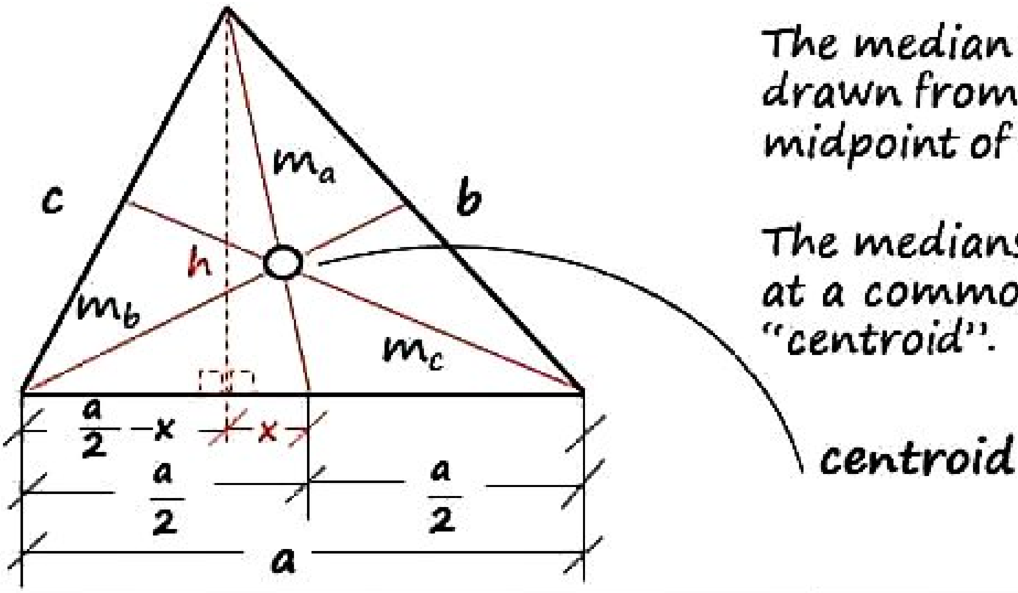
$$A_T = \tfrac{1}{2}(a)(a_a)$$

$$\boxed{a_a = \frac{2A_T}{a}}$$

$$\boxed{a_b = \frac{2A_T}{b}}$$

$$\boxed{a_c = \frac{2A_T}{c}}$$

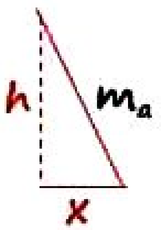
DERIVATION OF MEDIAN OF TRIANGLE



The median of a triangle is the line drawn from one vertex to the midpoint of its opposite side.

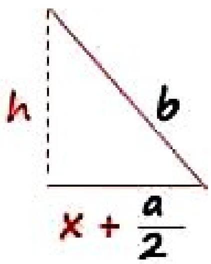
The medians of a triangle intersect at a common point called the "centroid".

For this triangle:



$$m_a^2 = h^2 + x^2$$

For this triangle:



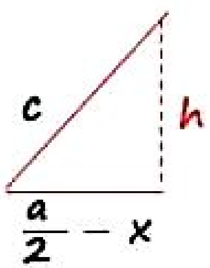
$$h^2 + (x + \frac{a}{2})^2 = b^2$$

$$h^2 + (x^2 + ax + \frac{a^2}{4}) = b^2$$

$$\underbrace{h^2 + (x^2}_{} + ax + \frac{a^2}{4}) = b^2$$

$$m_a^2 + ax + \frac{a^2}{4} = b^2$$

For this triangle:



$$h^2 + (\frac{a}{2} - x)^2 = c^2$$

$$h^2 + (\frac{a^2}{4} - ax + x^2) = c^2$$

$$m_a^2 + \frac{a^2}{4} - ax = c^2$$

$$b^2 + c^2 = m_a^2 + \frac{a^2}{4} + m_a^2 + \frac{a^2}{4}$$

$$b^2 + c^2 = 2m_a^2 + \frac{a^2}{2}$$

$$2m_a^2 = b^2 + c^2 - \frac{a^2}{2}$$

$$2m_a^2 = \frac{2b^2 + 2c^2 - a^2}{2}$$

$$m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4}$$

$$\boxed{m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}}$$

$$\boxed{m_b = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2}}$$

$$\boxed{m_c = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}}$$

where s, (semi perimeter) $= \frac{a+b+c}{2}$

DERIVE

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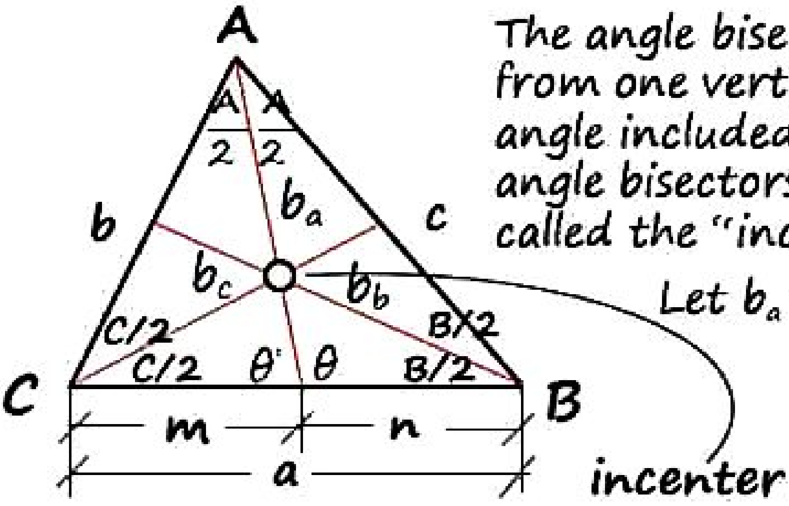
DERIVATION – LENGTH OF ANGLE BISECTOR OF TRIANGLE



The angle bisector of a triangle is the line drawn from one vertex to the opposite side bisecting the angle included between the other two sides. The angle bisectors of a triangle intersect at a point is called the "incenter".

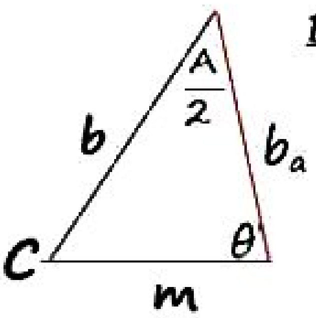
Let b_a = length of angle bisector for side a

$$\cos\theta' = -\cos\theta \quad \text{(Supplementary)}$$

$$a = m + n \longrightarrow n = a - m$$

$$s = (a + b + c)/2 = \text{semi perimeter}$$

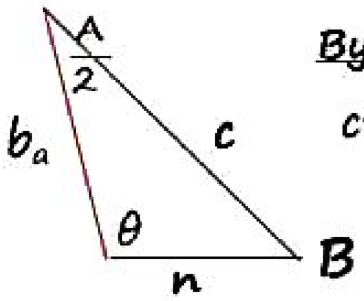
$$a + b + c = 2s$$



By cosine law:

$$b^2 = m^2 + b_a^2 - 2mb_a\cos\theta'$$

$$b^2 = m^2 + b_a^2 + 2mb_a\cos\theta \longrightarrow \text{Eq.1}$$



By cosine law:

$$c^2 = n^2 + b_a^2 - 2nb_a\cos\theta \longrightarrow \text{Eq.2}$$

Multiply Eq. 1 by n and Eq. 2 by m then add:

$$\text{Eq.2} \quad c^2m = n^2m + b_a^2m - \cancel{2mnb_a\cos\theta}$$

$$\text{Eq.1} + \quad b^2n = m^2n + b_a^2n + \cancel{2mnb_a\cos\theta}$$

$$b^2n + c^2m = m^2n + b_a^2n + n^2m + b_a^2m$$

$$b^2n + c^2m = b_a^2(m + n) + mn(m + n)$$

$$b^2n + c^2m = (m + n)(b_a^2 + mn)$$

$$b^2n + c^2m = (a)(b_a^2 + mn) \longrightarrow \text{Eq.3}$$

Derive m and n in terms of a, b and c:

By Angle Bisector Theorem:

$$\frac{m}{b} = \frac{n}{c}$$

$$m = \frac{nb}{c} \qquad m = \frac{b(a - m)}{c}$$

$$mc = ba - bm$$

$$mc + bm = ba$$

$$m(c + b) = ba$$

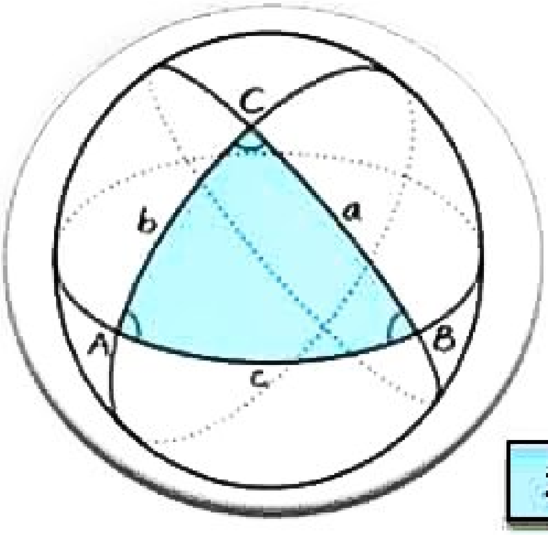
$$m = \frac{ab}{b+c} \;//$$

$$n = a - m = a - \frac{ab}{b+c}$$

$$n = \frac{ab + ac - ab}{b+c}$$

$$n = \frac{ac}{b+c} \;//$$

Spherical Trigonometry



Spherical Triangle

is a figure formed on the surface of a sphere by three great circular arcs intersecting pairwise in three vertices.

The sum of the interior angles is greater than 180° but less than 540°

$$180^\circ < (A + B + C) < 540^\circ$$

The area of a spherical triangle with radius R is:

$$A = \frac{\pi R^2 E}{180^\circ}$$

Where E is the spherical excess in degress and is given by:

$$E = A + B + C - 180^\circ$$

or

$$\tan\frac{E}{4} = \sqrt{\tan\frac{s}{2}\tan\frac{s-a}{2}\tan\frac{s-b}{2}\tan\frac{s-c}{2}}$$

where $s = \dfrac{(a + b + c)}{2}$

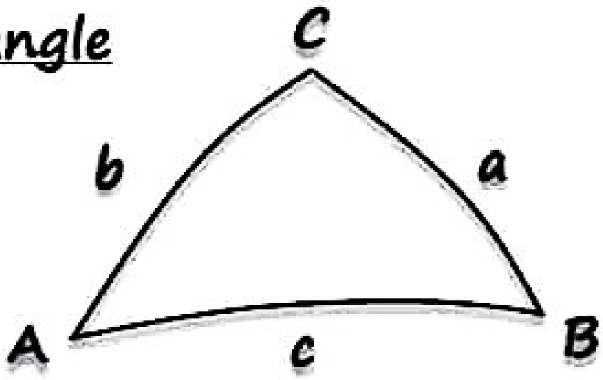
$$\text{Spherical Defect} = 360^\circ - (a + b + c)$$

In spherical trigonometry, earth is assumed to be a perfect sphere. One minute (0° 1') of arc from the center of the earth has a distance equivalent to one (1) nautical mile (6,080 feet) on the arc of great circle on the surface of the earth.

- 1 minute of arc = 1 nautical mile
- 1 nautical mile = 6080 feet
- 1 statute mile = 5280 feet
- 1 knot = 1 nautical mile per hour

Spherical Trigonometry

Oblique Spherical Triangle



Law of Sines

$$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C}$$

Law of Cosines for Sides

$$\cos a = (\cos b)(\cos c) + (\sin b)(\sin c)(\cos A)$$
$$\cos b = (\cos a)(\cos c) + (\sin a)(\sin c)(\cos B)$$
$$\cos c = (\cos a)(\cos b) + (\sin a)(\sin b)(\cos C)$$

Law of Cosines for Angles

$$\cos A = -(\cos B)(\cos C) + (\sin B)(\sin C)(\cos a)$$
$$\cos B = -(\cos A)(\cos C) + (\sin A)(\sin C)(\cos b)$$
$$\cos C = -(\cos A)(\cos B) + (\sin A)(\sin B)(\cos c)$$

Napier's Analogies

$$\frac{\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)} = \frac{\tan\left(\frac{a-b}{2}\right)}{\tan\left(\frac{c}{2}\right)}$$

$$\frac{\sin\left(\frac{a-b}{2}\right)}{\sin\left(\frac{a+b}{2}\right)} = \frac{\tan\left(\frac{A-B}{2}\right)}{\cot\left(\frac{C}{2}\right)}$$

$$\frac{\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)} = \frac{\tan\left(\frac{a+b}{2}\right)}{\tan\left(\frac{c}{2}\right)}$$

$$\frac{\cos\left(\frac{a-b}{2}\right)}{\cos\left(\frac{a+b}{2}\right)} = \frac{\tan\left(\frac{A+B}{2}\right)}{\cot\left(\frac{C}{2}\right)}$$

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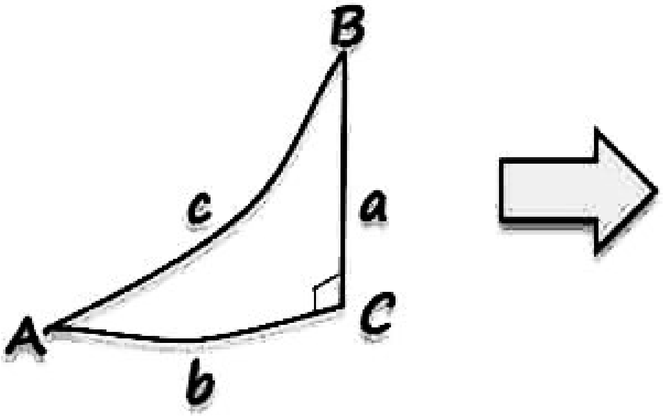
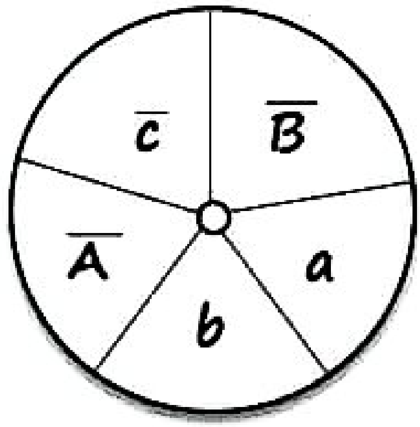
Spherical Trigonometry

Right Spherical Triangle





Napier's Circle

In Napier's Circle, the sides and angle of the triangle are written in consecutive order (not including the right angle), and complimentary angles are taken for quantities opposite the right angle.

Where:
$$\overline{A} = 90^{\circ} - A$$
$$\overline{B} = 90^{\circ} - B$$
$$\overline{c} = 90^{\circ} - c$$

Napier's Rules

sine – cosine opposite

SIN–COOP rule

The sine of any middle part is equal to the product of the cosines of its opposite parts.

If we take "a" as the middle part, its opposite parts are $\overline{c}$ and $\overline{A}$, then:

$$\sin a = (\cos\overline{c})(\cos\overline{A})$$
$$\sin a = \cos(90-c)\cos(90-A)$$
$$\sin a = (\sin c)(\sin A)$$

sine – tangent adjacent

SIN–TAAD rule

The sine of any middle part is equal to the product of the tangents of its adjacent parts.

If we take "$\overline{A}$" as the middle part, its adjacent parts are b and $\overline{c}$, then:

$$\sin\overline{A} = (\tan b)(\tan\overline{c})$$
or
$$\cos A = (\tan b)(\cot c)$$

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Pythagorean Triple Generator

$$a^2 + b^2 = (m^2 - n^2)^2 + (2mn)^2$$
$$= (m^4 - 2m^2n^2 + n^4) + 4m^2n^2$$
$$= (m^4 + 2m^2n^2 + n^4)$$
$$= (m^2 + n^2)^2$$
$$= (c)^2$$

Thus; $a^2 + b^2 = c^2$ ✓

<u>Example</u>

Given $m = 3$ and $n = 2$

$a = m^2 - n^2 = 3^2 - 2^2 = 5$
$b = 2mn = 2(3)(2) = 12$
$c = m^2 + n^2 = 3^2 + 2^2 = 13$

Thus $(a,b,c) = (5,12,13)$

$5^2 + 12^2 = 13^2$ ✓

(5,12,13) is a Phytagorean Triple!!

Given $m = 4$ and $n = 3$

$a = m^2 - n^2 = 4^2 - 3^2 = 7$
$b = 2mn = 2(4)(3) = 24$
$c = m^2 + n^2 = 4^2 + 3^2 = 25$

Thus $(a,b,c) = (7,24,25)$

$7^2 + 24^2 = 25^2$ ✓

(7,24,25) is a Phytagorean Triple!!

There are 16 Phytagorean Triples with $c \leq 100$

(a,b,c)
(3,4,5)
(5,12,13)
(8,15,17)
(7,24,25)
(20,21,29)
(12,35,37)
(9,40,41)
(28,45,53)
(11,60,61)
(16,63,65)
(33,56,65)
(48,55,73)
(13,84,85)
(36,77,85)
(39,80,89)
(65,72,97)

<u>Summary</u>

The Pythagorean Triple Generator!

$a = m^2 - n^2$
$b = 2mn$
$c = m^2 + n^2$

Such that a,b,c is a positive integer

$m > n > 0$

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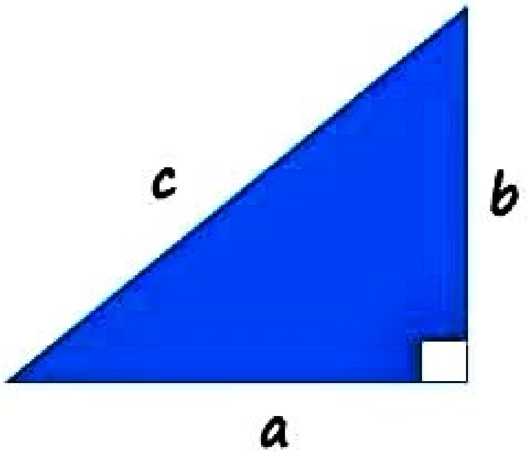
Pythagorean Triple Generator



A Phytagorean Triple consists of three positive integers a, b and c such that $a^2 + b^2 = c^2$. Such a triple is commonly written as (a, b, c) with the well known example of (3,4,5).

For instance, the triangle with sides $a = b = 1$ and $c = \sqrt{2}$ is a right triangle but $(1,1,\sqrt{2})$ is not a Phytagorean Triple because $\sqrt{2}$ is not an integer.

Derivation of the generator;

$$a^2 + b^2 = c^2 \longrightarrow b^2 = c^2 - a^2$$

$$b^2 = (c + a)(c - a)$$

<u>Divide both sides by (c−a)(b)</u>

$$\frac{b^2}{(c-a)(b)} = \frac{(c + a)(c - a)}{(c-a)(b)}$$

$$\frac{b}{(c-a)} = \frac{(c + a)}{(b)}$$

Set it equal to $\frac{m}{n}$

$$\text{Thus;}\quad \frac{b}{(c-a)} = \frac{(c + a)}{(b)} = \frac{m}{n}$$

$$\frac{b}{(c-a)} = \frac{m}{n} \longrightarrow \boxed{\frac{(c - a)}{b} = \frac{n}{m}}$$

$$\text{And;}\quad \boxed{\frac{(c + a)}{(b)} = \frac{m}{n}}$$

$$\frac{(c - a)}{b} = \frac{n}{m} \longrightarrow \frac{c}{b} - \frac{a}{b} = \frac{n}{m} \qquad \text{Eq.1}$$

$$\frac{(c + a)}{b} = \frac{m}{n} \longrightarrow \frac{c}{b} + \frac{a}{b} = \frac{m}{n} \qquad \text{Eq.2}$$

<u>Adding Equation 1 and 2:</u>

$$\frac{c}{b} = \frac{m^2 + n^2}{2mn}$$

<u>Thus, we may consider that:</u>

$$b = 2mn$$

and

$$c = m^2 + n^2$$

<u>Subtracting Eq.1 from Eq. 2</u>

$$\frac{a}{b} = \frac{m^2 - n^2}{2mn}$$

Since $b = 2mn$

Then $a = m^2 - n^2$

$$\left.\begin{aligned} a &= m^2 - n^2 \\ b &= 2mn \\ c &= m^2 + n^2 \end{aligned}\right\} \text{The Pythagorean Triple Generator!}$$

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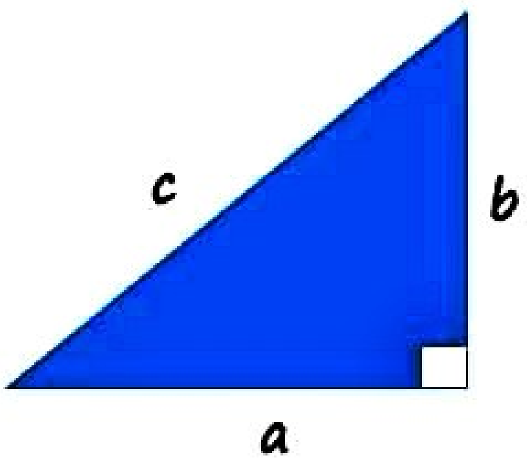
Pythagorean Triple Generator



A Phytagorean Triple consists of three positive integers a, b and c such that $a^2 + b^2 = c^2$. Such a triple is commonly written as (a, b, c) with the well known example of (3,4,5).

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Set it equal to $\frac{m}{n}$

Thus; $\frac{b}{(c-a)} = \frac{(c+a)}{(b)} = \frac{m}{n}$

$$\frac{b}{(c-a)} = \frac{m}{n} \longrightarrow \boxed{\frac{(c-a)}{b} = \frac{n}{m}}$$

And; $\boxed{\frac{(c+a)}{(b)} = \frac{m}{n}}$

$$\frac{(c-a)}{b} = \frac{n}{m} \longrightarrow \frac{c}{b} - \frac{a}{b} = \frac{n}{m} \quad \text{Eq.1}$$

$$\frac{(c+a)}{b} = \frac{m}{n} \longrightarrow \frac{c}{b} + \frac{a}{b} = \frac{m}{n}$$

Eq.2

Adding Equation 1 and 2:

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Thus, we may consider that:

$$b = 2mn$$

and

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Subtracting Eq.1 from Eq. 2

$$\frac{a}{b} = \frac{m^2 - n^2}{2mn}$$

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Then $a = m^2 - n^2$

$$\left.\begin{aligned} a &= m^2 - n^2 \\ b &= 2mn \\ c &= m^2 + n^2 \end{aligned}\right\} \text{The Pythagorean Triple Generator!}$$